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Perturbations of Attractors of Differential Equations*

VICTOR A. PLISS

Faculty of Mathematics and Mechanics, Leningrad State University, Leningrad, USSR

AND

GEORGE R. SELL

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

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In this paper we study small C^1-perturbations of a differential equation that has a hyperbolic attractor $\mathscr{K}$. We show that if $\mathscr{K}$ has a suitable Lipschitz property and if the perturbation is small enough, then there is a homeomorphism $h: \mathscr{K} \to \mathscr{K}^Y$, where $\mathscr{K}^Y$ is a hyperbolic attractor for the perturbed equation. Examples are included.

1. INTRODUCTION

In recent years researchers in differential equations have devoted considerable efforts to the study of methods of reducing high dimensional problems to low dimensional problems. These efforts have been both on the theoretical level, where for example, one tries to use the techniques of ordinary differential equations to study problems arising in partial differential equations, and on the practical level, say in numerical analysis, where one tries to reduce problems involving systems of thousands (or millions) of ordinary equations to an equivalent problem for a handful of such equations. In the area of the long-time dynamics of differential equations, these

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recent studies have been an outgrowth of the Reduction Principle paradigm; see Pliss [13].

The Reduction Principle, as it can be applied to the system of equations

$$p' = F(p, q), \qquad q' = G(p, q), \tag{1.1}$$

where $p \in \mathscr{P}$, $q \in \mathscr{Q}$, $F: \mathscr{P} \times \mathscr{Q} \to \mathscr{P}$ and $G: \mathscr{P} \times \mathscr{Q} \to \mathscr{Q}$ are C^1-functions and $\mathscr{P}$ and $\mathscr{Q}$ are Banach spaces, takes on the following form. Assume that the surface $M = \text{graph}\, \phi$ is invariant and exponentially stable for (1.1), where $\phi: \mathscr{P} \to \mathscr{Q}$ is a Lipschitz continuous function. In this case the long-time dynamics of the system (1.1) is completely described by the solutions of the reduced system

$$p' = F(p, \phi(p)), \qquad p \in \mathscr{P}.$$

The center manifold theory, as well as the theory of inertial manifolds, is an illustration of the Reduction Principle; see for example, Carr [2] Foias, Sell, and Temam [4], Henry [6], and Mallet-Paret and Sell [8].

The theory of approximate inertial manifolds is based on a variation of the procedure described above. In order to explain this we recall that $M = \text{graph}\ \phi$ is an invariant set for (1.1) if and only if $dq/dt = \nabla\phi\, dp/dt$, where $q = \phi(p)$. In other words one has

$$\nabla\phi(p)\, F(p, \phi(p)) = G(p, \phi(p)), \qquad \text{a.a. } p \in \mathscr{P}.$$

As a result, for any given Lipschitz continuous mapping $q = \phi_a(p)$, the manifold $\mathscr{M}_a = \text{graph}\ \phi_a$ is an invariant set for the perturbed system

$$p' = F(p, q), \qquad q' = G(p, q) + E(p) \tag{1.2}$$

if and only if

$$E(p) = \nabla\phi_a(p)\, F(p, \phi_a(p)) - G(p, \phi_a(p)), \qquad \text{a.a. } p \in \mathscr{P}.$$

If in addition, the manifold M_a is exponentially stable for (1.2), then the long-time dynamics of (1.2) is completely described by the solutions of the reduced equation

$$p' = F(p, \phi_a(p)). \tag{1.3}$$

The term E appearing in (1.2) can be viewed as an *error term*. The manifold $\mathscr{M}_a$ is an invariant manifold for (1.1) if and only if $E(p) \equiv 0$. By introducing a suitable norm one can use $\|E\|$ to measure how far $\mathscr{M}_a$ is from being an invariant manifold. As $\|E\|$ gets smaller, $\mathscr{M}_a$ gets closer to being invariant.

This paradigm for approximate inertial manifolds is fully justified for such infinite dimensional dynamical systems as the two-dimensional

Navier–Stokes equations (2DNS) on a suitable bounded region $\Omega \subset R^2$; see Foias and Temam [5], Marion and Temam [9], Sell [17], Temam [19], and Titi [20]. (In this setting $\mathscr{P}$ is a finite dimensional space and $\mathscr{Q}$ is infinite dimensional.) It is because of the fact that for the 2DNS the error term E in (1.2) can be made small, in an appropriate norm, that one can hope to see the long-time dynamics of (1.1) being well-approximated by the solutions of (1.2), or equivalently, by the solutions of the reduced equation (1.3).

We emphasize here that we are interested in the theory of approximating the long-time dynamics of a given differential equation. This theory, which we call *Aprroximation Dynamics*, differs significantly from the classical approximation theories wherein one seeks approximate solutions for a given differential equation on a finite time interval $0 \leqslant t \leqslant T$, where $T < \infty$.

The theory of Approximation Dynamics is of interest in the study of infinite dimensional systems like the Navier–Stokes equations as well as in the study of finite dimensional systems of ordinary differential equations (ODE). We emphasize that the theory of Approximation Dynamics for ODEs, as well as the associated theories of approximate inertial manifolds, plays a central role in the numerical integration of partial differential equations. Typical space discretizations of the 2DNS, for example, oftentimes lead to systems of thousands of ODEs.

In this paper we study two systems of ODE,

$$\frac{dx}{dt} = X(x) \tag{1.4}$$

and

$$\frac{dy}{dt} = X(y) + Y(y), \tag{1.5}$$

where $x, y \in R^n$ and X and Y are C^1-functions from R^n into itself. For any $x_0 \in R^n$ we shall let $x(t, x_0)$ (or $y(t, x_0)$) denote the maximally defined solution of (1.4) (or (1.5)) that satisfies $x(0, x_0) = y(0, x_0) = x_0$.

In order to simplify the notation we shall assume that the perturbation term Y is bounded in the C^1-norm and satisfies

$$\| Y \|_{C^1} \leqslant \delta \tag{1.6}$$

for a suitable $\delta > 0$.

Let $\mathscr{K}$ denote a given compact invariant set for (1.4). For each $x_0 \in \mathscr{K}$ we let $\Phi(t, x_0)$ denote the fundamental operator solution of the linear system

$$\frac{dx}{dt} = \frac{\partial X(x(t, x_0))}{\partial x} x \tag{1.7}$$

that satisfies $\Phi(0, x_0) = I$, where I is the identity operator on R^n. We shall say that the linear system (1.7) is *hyperbolic along* $x(t, x_0)$ *on an interval* J *with constants* a, λ_1, and λ_2 provided $\lambda_2 < \lambda_1$, $\lambda_1 > 0$, $a \geqslant 1$ and there exist complementary linear spaces $\mathscr{U}^n(t, x_0)$ and $\mathscr{U}^s(t, x_0)$, $\dim \mathscr{U}^n = k$, $\dim \mathscr{U}^s = n - k$, and

$$\Phi(t, x_0)\,\mathscr{U}^s(0, x_0) = \mathscr{U}^s(t, x_0), \qquad \Phi(t, x_0)\,\mathscr{U}^n(0, x_0) = \mathscr{U}^n(t, x_0)$$

for all $t \in J$. Furthermore if $\bar{x} \in \mathscr{U}^s(\tau, x_0)$, then

$$|\Phi(t, x_0)\,\Phi^{-1}(\tau, x_0)\,\bar{x}| \leqslant a\,|\bar{x}|\,e^{-\lambda_1(t-\tau)}, \qquad \text{for} \quad t \geqslant \tau,\ t, \tau \in J, \tag{1.8}$$

and if $\bar{x} \in \mathscr{U}^n(\tau, x_0)$, then

$$|\Phi(t, x_0)\,\Phi^{-1}(\tau, x_0)\,\bar{x}| \leqslant a\,|\bar{x}|\,e^{-\lambda_2(t-\tau)}, \qquad \text{for} \quad t \leqslant \tau,\ t, \tau \in J. \tag{1.9}$$

We shall say that a compact, invariant set $\mathscr{K}$ for (1.4) is a *hyperbolic attractor* if $\mathscr{K}$ satisfies the following two properties:

(1) The linear system (1.7) is hyperbolic along $x(t, x_0)$ on R with constants a, λ_1, and λ_2 for every $x_0 \in \mathscr{K}$.

(2) There exists an $r > 0$ such that for each $x_0 \in \mathscr{K}$ there exists a k-dimensional locally invariant disk $\mathscr{D}(x_0) \subset \mathscr{K}$ with the center at the point x_0 and radius r such that if $x \in \mathscr{D}(x_0)$ then at the point x the disk $\mathscr{D}(x_0)$ is tangent to the linear space $\mathscr{U}^n(0, x)$.

A hyperbolic attractor $\mathscr{K}$ is said to satisfy the *Lipschitz property* if the mapping $x_0 \to \mathscr{U}^n(0, x_0)$ is a Lipschitz continuous mapping of $\mathscr{K}$ into the Grassmanian of k-dimensional subspaces of R^n. The notion of hyperbolicity introduced here has some antecedents in the literature; see for example, Fenichel [3], Hirsch, Pugh, and Shub [7], Meyer and Sell [10], Sacker [15], Sell [16], and Smale [18]. In the sequel we shall let $\mathscr{K}$ denote a given hyperbolic attractor for (1.4) that satisfies the Lipschitz property.

The Lipschitz property for the neutral manifolds $\mathscr{U}^n(0, x_0)$ plays an important role in the theory we present here. As we shall see later the Lipschitz property implies that the disks $\mathscr{D}(x_0)$ described in item (2) above are uniquely determined. Examples of hyperbolic attractors that satisfy the Lipschitz property are given in Section 6.

Let for $x_0 \in \mathscr{K}$ we define the sets $\mathscr{S}_1(x_0)$, $\mathscr{S}_2(x_0)$, $\mathscr{S}_3(x_0)$, ..., $\mathscr{S}(x_0)$ by

$$\mathscr{S}_1(x_0) = \bigcup_{x \in \mathscr{D}(x_0)} \mathscr{D}(x), \qquad \mathscr{S}_{i+1}(x_0) = \bigcup_{x \in \mathscr{S}_i(x_0)} \mathscr{D}(x), \qquad \text{for} \quad i \geqslant 1,$$

and

$$\mathscr{S}(x_0) = \bigcup_{i=1}^{\infty} \mathscr{S}_i(x_0).$$

Note that the set $\mathscr{S}(x_0)$ is invariant and $\mathscr{S}(x_0) \subset \mathscr{K}$, for every $x_0 \in \mathscr{K}$. Furthermore, if $x_1 \in \mathscr{S}(x_0)$, then $\mathscr{S}(x_1) = \mathscr{S}(x_0)$, because of the uniqueness of the disks. This implies that if $\mathscr{S}(x_0) \cap \mathscr{S}(x_1)$ is nonempty, then $\mathscr{S}(x_0) = \mathscr{S}(x_1)$. The set $\mathscr{S}(x_0)$ is referred to as the *leaf* of $\mathscr{K}$ through x_0.

The principle objective of this paper is to prove the following result:

MAIN THEOREM. *Let $\mathscr{K}$ be a given hyperbolic attractor of* (1.4) *that satisfies the Lipschitz property. For every $\varepsilon > 0$ there is a $\delta > 0$ such that the Y satisfies* (1.6) *for this δ, then there is a one-to-one, continuous mapping*

$$h : \mathscr{K} \to R^n$$

that satisfies $|h(x) - x| \leqslant \varepsilon$, and the image $\mathscr{K}^Y = h(\mathscr{K})$ is a compact, invariant set for (1.5). *Furthermore for each leaf $\mathscr{S} \subset \mathscr{K}$, the restriction $h|_{\mathscr{S}}$ is a locally Lipschitz continuous mapping of $\mathscr{S}$ onto a leaf $\mathscr{S}^Y \subset \mathscr{K}^Y$. In addition, $\mathscr{K}^Y$ is a hyperbolic attractor for* (1.5).

Remark. The inequality (1.6) is stated in a simplified form. What is really used in the proof of the Main Theorem is that

$$\max(\| Y(y) \|, \| DY(y) \|) \leqslant \delta, \qquad y \in U(\mathscr{K}, \beta),$$

where $U(\mathscr{K}, \beta)$ is a fixed β-neighborhood of $\mathscr{K}$.

This paper is organized as follows. In Section 2 we show that the disks $\mathscr{D}(x_0)$ are uniquely determined when $\mathscr{K}$ satisfies the Lipschitz property. In Section 3 we present the basic construction of the mapping $h : \mathscr{K} \to R^n$. We show here that h is a locally Lipschitz continuous mapping of each leaf $\mathscr{S}(x_0)$ of $\mathscr{K}$ onto an invariant set $\mathscr{S}^Y(h(x_0))$ for (1.5) and that $|h(x) - x| \leqslant \varepsilon$ for all $x \in \mathscr{K}$. We also show that h is continuous as a mapping of $\mathscr{K}$ into R^n. In Section 4 we argue that the image set $\mathscr{K}^Y = h(\mathscr{K})$ is an hyperbolic attractor for (1.5). The proof of the Main Theorem is completed in Section 5 wherein we show that h is one-to-one whenever ε is sufficiently small. Examples are included in Section 6.

Let $\mathscr{U}^s(x_0) = \mathscr{U}^s(0, s_0)$, $\mathscr{U}^n(x_0) = \mathscr{U}^n(0, x_0)$. It is clear that $\mathscr{U}^s(t, x_0) = \mathscr{U}^s(x(t, x_0))$ and $\mathscr{U}^n(t, x_0) = \mathscr{U}^n(x(t, x_0))$ for all t. Let $\mathscr{N}(x)$, $x \in \mathscr{K}$ be the $(n-k)$-dimensional hyperplane perpendicular to $\mathscr{U}^n(x)$ at the point x.

2. UNIQUENESS OF DISKS

Let $\mathscr{K}$ be a given hyperbolic attractor for (1.4) that satisfies the Lipschitz property, and let L be the Lipschitz constant for the mapping $x_0 \to \mathscr{U}^n(0, x_0)$. We claim that if $\mathscr{D}_1(x_0)$ and $\mathscr{D}_2(x_0)$ are two disks that satisfy the condition (2) in the definition of a hyperbolic attractor, then $\mathscr{D}_1(x_0)=\mathscr{D}_2(x_0)$.

In order to prove this, let us move the origin to the point x_0 and rotate the coordinate axes, to obtain a new coordinate system (u, v), where u is a k-dimensional vector and v is an $(n-k)$-dimensional vector. We assume that the coordinate axes have been fixed so that the space $v=0$ coincides with the space $\mathscr{U}^n(x_0)$. The disks $\mathscr{D}_1(x_0)$ and $\mathscr{D}_2(x_0)$ can then be represented in the form

$$\mathscr{D}_1(x_0)=\{v=f(u): |u|\leqslant r\}, \qquad \mathscr{D}_2(x_0)=\{v=g(u): |u|\leqslant r\}.$$

Let u_1 be an arbitrary k-dimensional vector with $|u_1|<r$ and let $w(s)=f(su_1)-g(su_1)$, $0\leqslant s\leqslant 1$. Then

$$\frac{dw}{ds}=\left(\frac{\partial f}{\partial u}-\frac{\partial g}{\partial u}\right)u_1,$$

and

$$\frac{d\,|w|}{ds}\leqslant\left|\frac{dw}{ds}\right|\leqslant\left|\frac{\partial f}{\partial u}-\frac{\partial g}{\partial u}\right|\,|u_1|.$$

From the Lipschitz property one has $|\partial f/\partial u-\partial g/\partial u|\leqslant L\,|f(su_1)-g(su_1)|$ $=L\,|w|$, and consequently

$$\frac{d\,|w|}{ds}\leqslant L\,|u_1|\,|w|.$$

After integrating this inequality on the interval $0\leqslant s\leqslant 1$ we find $|w(1)|\leqslant$ $|w(0)|\,e^{L|u_1|}$. Since $w(0)=0$, we have $w(1)=0$, or $f(u_1)=g(u_1)$ and $\mathscr{D}_1(x_0)=\mathscr{D}_2(x_0)$.

3. Existence of h

In this section we first prove the existence of a continuous mapping $h\colon \mathscr{K}\to R^n$ such that the image $\mathscr{K}^Y=h(\mathscr{K})$ is a compact, invariant set for (1.5). In the following sections we show that $\mathscr{K}^Y$ is a hyperbolic attractor, and that h is a homeomorphism.

THEOREM 1. *For each* $\varepsilon>0$ *there exists a* $\delta>0$ *such that if* Y *satisfies* (1.6) *with this* δ, *then the system* (1.5) *has an invariant set* $\mathscr{K}^Y$ *and there exists a continuous mapping* h *of* $\mathscr{K}$ *onto* $\mathscr{K}^Y$ *such that* $h(x)\in\mathscr{N}(x)\cap\mathscr{K}^Y$ *and* $|h(x)-x|\leqslant\varepsilon$, *for all* $x\in\mathscr{K}$. *Furthermore, for each leaf* $\mathscr{S}\subset\mathscr{K}$, *the* *restriction* $h|_{\mathscr{S}}$ *is a locally Lipschitz continuous mapping of* $\mathscr{S}$ *into* $\mathscr{K}^Y$. *In addition there is an induced continuous flow* $\hat{S}(t)\,x$ *on* $\mathscr{K}$ *such that*

$$h(\hat{S}(t)\,x) = y(t, h(x)), \qquad \text{for all } x \in \mathscr{K} \text{ and } t \in R. \tag{3.1}$$

Proof. Define σ by $11\sigma = \min(\lambda_1, \lambda_1 - \lambda_2)$, where a, λ_1, and λ_2 are given by (1.8) and (1.9). Next we note that there exists an α, $0 < \alpha \leqslant \pi/2$, such that the angle $\angle\,(\mathscr{U}^s(x_0), \mathscr{U}^n(x_0))$ satisfies

$$\angle\,(\mathscr{U}^s(x_0), \mathscr{U}^n(x_0)) \geqslant \alpha, \qquad x_0 \in \mathscr{K}. \tag{3.2}$$

Let T be a positive number satisfying both

$$e^{-(\lambda_1 - \sigma)T} \leqslant \frac{\sin(\alpha/2)}{6a}, \qquad e^{-(\lambda_1 - 3\sigma)T} \leqslant \frac{3}{16a}. \tag{3.3}$$

Next fix c, $0 < c \leqslant 1/2$, so that whenever the vector ζ is satisfying

$$\angle\,(\mathscr{U}^n(x_0), \zeta) \leqslant c\alpha, \tag{3.4}$$

then

$$\angle\,(\mathscr{U}^n(x(t, x_0)), \Phi(t, x_0)\,\zeta) \leqslant \frac{\alpha}{4}, \qquad 0 \leqslant t \leqslant 2T, \tag{3.5}$$

and

$$\angle\,(\mathscr{U}^n(x(t, x_0)), \Phi(t, x_0)\,\zeta) \leqslant \frac{1}{4}\,c\alpha, \qquad T \leqslant t \leqslant 2T, \tag{3.6}$$

where as earlier $\Phi(t, x_0)$ is a fundamental operator solution of the system (1.7).

With c given by (3.4) we fix $r > 0$, where r is the radius of the disk $\mathscr{D}(x)$ in $\mathscr{K}$, so that if x_0, x_1 are distinct points in some leaf $\mathscr{S}$ with $|x_0 - x_1| \leqslant r$, then one has

$$\angle\,(\mathscr{U}^n(x_0), x_0 - x_1) \leqslant c\alpha.$$

In accordance with Perron's Stable Manifold Theorem, for each $x_0 \in \mathscr{K}$ there exists $(n-k)$-dimensional disk $\mathscr{D}^p(x_0)$ (the so called Perron disk) such that, if $x_1 \in \mathscr{D}^p(x_0)$, then

$$|x(t, x_1) - x(t, x_0)| \leqslant 2a\,|x_1 - x_0|\,e^{-(\lambda_1 - \sigma)t}, \qquad \text{for} \quad t \geqslant 0, \tag{3.7}$$

where σ is given above. The radius b of the Perron disk $\mathscr{D}^p(x_0)$ does not depend on x_0, and $\mathscr{D}^p(x_0)$ is arbitrarily close to the linear space $\mathscr{U}^s(x_0)$ provided b is sufficiently small. Finally the disk $\mathscr{D}^p(x_0)$ depends continuously on $x_0 \in \mathscr{K}$.

Next we will choose β, $0<\beta<\min(r, b)$, so that in $U(x(t, x_0), \beta)$, the β-neighborhood of the point $x(t, x_0)$, the linear system (1.7) *dominates* the nonlinearity in (1.4). Recall that if β is small enough, then the Perron disks form a foliation of the neighborhood $U(x_0, \beta)$; i.e., one has

$$U(x_0, \beta) \subset \bigcup_{x_1 \in \mathscr{D}(x_0)} \mathscr{D}^p(x_1), \qquad \text{for all} \quad x_0 \in \mathscr{K}; \tag{3.8}$$

see Pliss [13, 14]. For any $x_0 \in \mathscr{K}$ we define

$$\Gamma(x_0, \beta) \stackrel{\text{def}}{=} \{x+y: x \in \mathscr{D}(x_0), y \in \mathscr{N}(x), |y| \leqslant \beta\}.$$

Because of the Lipschitz property, it follows, from the Tubular Neighborhood Theorem, that there is a small $\beta>0$ such that for any $x_0 \in \mathscr{K}$ there is a Lipschitz continuous mapping

$$\phi = \phi_{x_0}: \Gamma(x_0, \beta) \to \mathscr{D}(x_0), \qquad s_0 \in \mathscr{K} \tag{3.9}$$

that satisfies

$$y \in \mathscr{N}(\phi(y)), \qquad y \in \Gamma(x_0, \beta). \tag{3.10}$$

While the mapping ϕ_{x_0} depends on x_0, we note that if $x_1 \in \mathscr{D}(x_0)$, then ϕ_{x_0} and ϕ_{x_1} agree on the set $\Gamma(x_0, \beta) \cap \Gamma(x_1, \beta)$. Also the Lipschitz constant $\bar{L}$ for ϕ_{x_0} is independent of x_0.

Let x_1 and y_0 be such points that $x_1 \in \mathscr{K}$ and $|y_0 - x_1| \leqslant \beta$. It follows from (3.8) that there is an $x_0 \in \mathscr{D}(x_1)$ such that $y_0 \in \mathscr{D}^p(x_0)$. Since $\mathscr{D}^p(x_0)$ is arbitrarily close to $\mathscr{U}^s(x_0)$, it follows from (3.2) that

$$|x_0 - y_0| \leqslant \frac{|y_0 - x_1|}{\sin(2\alpha/3)}. \tag{3.11}$$

For the remainder of the argument we fix $\beta>0$ so that (3.8)–(3.11) hold.

We now turn to the nonlinear unperturbed equation (1.4). We use the continuity of solutions of (1.4) to fix a $\bar{c}$, $0<\bar{c}\leqslant 1$, so that the following four properties hold: First, whenever $x_1 \in \mathscr{K}$ and y_0 satisfies $|y_0 - x_1| \leqslant \bar{c}\beta$, then

$$x(t, y_0) \in \Gamma((x(t, x_0), \beta), \qquad 0 \leqslant t \leqslant 2T, \tag{3.12}$$

where $x_0 \in \mathscr{D}(x_1)$ is chosen so that $y_0 \in \mathscr{D}^p(x_0)$. Furthermore, for sufficiently small $\bar{c}$ the linear system (1.7) is hyperbolic along $x(t, y_0)$ on the interval $0 \leqslant t \leqslant 2T$ with constants $2a$, $\lambda_1 - \sigma$, and $\lambda_2 + \sigma$, This implies that the mappings $x_0 \to \mathscr{U}^n(x_0), \mathscr{U}^s(x_0)$ have continuous extensions from $\mathscr{K}$ to

$U(\mathscr{K}, \bar{c}\beta)$, the $\bar{c}\beta$-neighborhood of $\mathscr{K}$. As a second condition on $\bar{c}$ we require that

$$\angle\,(\mathscr{U}^n(x_0), \mathscr{U}^n(y_0)) \leqslant \frac{c\alpha}{8}, \qquad x_0 \in \mathscr{K},\ y_0 \in \mathscr{N}(x_0),\ |\,y_0| \leqslant \bar{c}\beta. \quad (3.13)$$

Third, we ask that $\bar{c}$ be chosen so that if $x_0 \in \mathscr{K}$, $|x_0 - x_i| \leqslant \bar{c}\beta$, $i = 1, 2$, and the angle satisfies $\angle\,(\mathscr{U}^n(x_0), x_1 - x_2) \leqslant c\alpha$, then one has

$$\angle\,(\mathscr{U}^n(x(t, x_0)), x(t, x_1) - x(t, x_2)) \leqslant \frac{\alpha}{3}, \qquad 0 \leqslant t \leqslant 2T,$$

and

$$\angle\,(\mathscr{U}^n(x(t, x_0)), x(t, x_1) - x(t, x_2)) \leqslant \frac{1}{2}c\alpha, \qquad T \leqslant t \leqslant 2T.$$

Let y_0 satisfy $|\,y_0 - x_1| \leqslant \bar{c}\beta$ for some $x_1 \in \mathscr{K}$, and let $y_0 \in \mathscr{D}^p(x_0)$, where $x_0 \in \mathscr{D}(x_1)$. By extending the mapping ϕ given by (3.9), it follows from (3.10) and (3.12) that

$$x(t, y_0) \in \mathscr{N}(\phi(x(t, y_0))), \qquad 0 \leqslant t \leqslant 2T,$$

and from (3.7) and (3.11) one obtains

$$|x(t, y_0) - \phi(x(t, y_0))| \leqslant |x(t, x_0) - x(t, y_0)| \leqslant \frac{2a\,|\,y_0 - x_1|}{\sin(2\alpha/3)}\, e^{-(\lambda_1 - \sigma)t}, \quad (3.14)$$

for $0 \leqslant t \leqslant 2T$. From (3.3) and (3.14) we get

$$|x(t, y_0) - \phi(x(t, y_0))| \leqslant \frac{|\,y_0 - x_1|}{3}, \qquad T \leqslant t \leqslant 2T. \quad (3.15)$$

Fourth, we ask that $\bar{c}$ satisfy $\bar{c} \leqslant 3(\bar{L} + 3)^{-1}$, where $\bar{L}$ is the Lipschitz constant for ϕ.

With T, c, r, β, and $\bar{c}$ so chosen, we let ε be given so that $0 < \varepsilon \leqslant \bar{c}\beta$. We then pick $\delta > 0$ so that the following four properties hold for Eqs. (1.5) and the linear system

$$\frac{dy}{dt} = \frac{\partial(X(y(t, y_0)) + Y(y(t, y_0)))}{\partial y}\, y, \quad (3.16)$$

where (1.6) is satisfied with this δ:

(I) If $|\,y_0 - x| \leqslant \varepsilon$ for some $x \in \mathscr{K}$, then

$$|\,y(t, y_0) - x(t, y_0)| \leqslant \frac{\varepsilon}{6}, \qquad 0 \leqslant t \leqslant 2T. \quad (3.17)$$

(II) If $x_0 \in \mathscr{K}$ and if $|y_0 - x_0| \leqslant \varepsilon$, then

$$|y(t, y_0) - x(t, x_0)| \leqslant \frac{\beta}{2}, \qquad 0 \leqslant t \leqslant 2T. \tag{3.18}$$

(III) Next we consider next the linear system (3.16). Let $\Psi(t, y_0)$ denote the fundamental operator solution of the system (3.16) that satisfies $\Psi(0, y_0) = I$. We note that if $x_0 \in \mathscr{K}$ and $|x_0 - y_0| \leqslant \bar{c}\beta$, then for sufficiently small δ the linear system (3.16) is hyperbolic along $y(t, y_0)$ on the interval $0 \leqslant t \leqslant 2T$ with constants $3a$, $\lambda_1 - 2\sigma$, and $\lambda_2 + 2\sigma$. These constants are independent of x_0 and y_0. Let $\mathscr{U}^n_Y(y(t, y_0))$ and $\mathscr{U}^s_Y(y(t, y_0))$ denote the associate neutral and stable spaces for $0 \leqslant t \leqslant 2T$. The linear spaces $\mathscr{U}^n_Y(y_0)$ and $\mathscr{U}^s_Y(y_0)$ can be made to be close to the corresponding linear spaces $\mathscr{U}^n(y_0)$ and $\mathscr{U}^s(y_0)$ for (1.4) by choosing δ sufficiently small. As a further condition on δ we require that

$$\angle\,(\mathscr{U}^n_Y(y_0), \mathscr{U}^n(y_0)) \leqslant \frac{c\alpha}{8}, \qquad y_0 \in U(\mathscr{K}, \bar{c}\beta). \tag{3.19}$$

In addition for small δ, it follows from (3.5) and (3.6) that if the vector η is such that $\angle\,(\mathscr{U}^n(x_0), \eta) \leqslant c\alpha$, then

$$\angle\,(\mathscr{U}^n(x(t, x_0)),\ \Psi(t, y_0)\,\eta) \leqslant \tfrac{2}{5}\alpha, \qquad 0 \leqslant t \leqslant 2T, \tag{3.20}$$

and

$$\angle\,(\mathscr{U}^n(x(t, x_0)),\ \Psi(t, y_0)\,\eta) \leqslant \tfrac{3}{5}c\alpha, \qquad T \leqslant t \leqslant 2T. \tag{3.21}$$

(IV) Let x_i and y_i, $i = 0, 1$, be points that satisfy $x_0 \in \mathscr{K}$, $x_1 \in \mathscr{D}(x_0)$, $y_i \in \mathscr{N}(x_i)$, $i = 0, 1$, $|x_1 - x_0| \leqslant \varepsilon$, $|y_1 - y_0| \leqslant \varepsilon$, and $|y_1 - x_1| \leqslant \varepsilon$. From (3.20) and (3.21) it follows that if δ is sufficiently small and

$$\angle\,(\mathscr{U}^n_Y(y_0), y_1 - y_0) \leqslant c\alpha,$$

then

$$\angle(\mathscr{U}^n_Y(y(t, y_0)), y(t, y_1) - y(t, y_0)) \leqslant \frac{\alpha}{2}, \qquad 0 \leqslant t \leqslant 2T,$$

and

$$\angle\,(\mathscr{U}^n_Y(y(t, y_0)), y(t, y_1) - y(t, y_0)) \leqslant \frac{3}{4}c\alpha, \qquad T \leqslant t \leqslant 2T. \tag{3.22}$$

Now let $x_1 \in \mathscr{K}$ and y_0 be chosen so that $|x_0 - y_1| \leqslant \varepsilon$. From (3.18) we see that

$$y(t, y_0) \in \Gamma\left(x_0, \frac{\beta}{2}\right), \qquad 0 \leqslant t \leqslant 2T,$$

and that

$$y(t, y_0) \in \mathcal{N}(\phi(y(t, y_0))), \qquad 0 \leqslant t \leqslant 2T,$$

as well as

$$| y(t, y_0) - \phi(y(t, y_0))| \leqslant \frac{\beta}{2}, \qquad 0 \leqslant t \leqslant 2T.$$

Using the Lipschitz constant $\bar{L}$ for ϕ together with (3.15) and (3.17), we obtain

$$| y(t, y_0) - \phi(y(t, y_0))| \leqslant K\varepsilon, \qquad T \leqslant t \leqslant 2T, \tag{3.23}$$

where $K = (1/6)(\bar{L} + 3)$. (Note that K depends only on $\mathcal{K}$.)

Let $\mathscr{E}$ denote the collection of (w, x) in $\mathcal{K} \times \mathcal{K}$ such that $w \in \mathcal{K}$ and $x \in \mathscr{S}(w)$. Let Q denote the collection of all functions $f: \mathscr{E} \to R^n$ that satisfy:

(1) f is continuous.

(2) $f(w, x) \in \mathcal{N}(x)$ for all $(w, x) \in \mathscr{E}$.

(3) $|f(w, x)| \leqslant K\varepsilon$ for all $(w, x) \in \mathscr{E}$.

(4) For every $(w, x_1) \in \mathscr{E}$ there is an $\varepsilon_1 = \varepsilon_1(w, x_1, f)$, $0 < \varepsilon_1 \leqslant \varepsilon$ such that

$$| f(w, x_1) - f(w, x_2)| \leqslant \ell \, | x_2 - x_2 |$$

for all $(w, x_2) \in \mathscr{E}$ with $| x_1 - x_2 | \leqslant \varepsilon_1$, where $\ell \leqslant \tan((3/4)\, c\alpha) < 1$.

(5) $f(w_1, x_1) = f(w_2, x_2)$ whenever (w_1, x_1), $(w_2, x_2) \in \mathscr{E}$ and $x_1 = x_2$.
Introduce in Q the usual metric:

$$d(f_1, f_2) = \sup_{(w, x) \in \mathscr{E}} | f_1 - f_2 |. \tag{3.24}$$

Since Q is a closed subset of $C^0(\mathscr{E}, R^n) \cap L^\infty(\mathscr{E}, R^n)$, Q is complete with this metric.

Property (5) is equivalent to saying that $f(w, x)$ is independent of w, provided $w \in \mathscr{S}(x)$. As a result, if $f \in Q$, then one has

$$f(w, x) = f(x, x), \qquad \text{for all } x \in \mathcal{K},\ w \in \mathscr{S}(x).$$

In addition the mapping $x \to f(x, x)$ is a continuous mapping of $\mathcal{K}$ into R^n.

Let $f \in Q$ and $w \in \mathcal{K}$ be fixed. For $x_1 \in \mathscr{S}(w)$, let $y_1 = x_1 + f(w, x_1)$. For $0 \leqslant \tau \leqslant 2T$ we define $\bar{f}_\tau = A_\tau f$ by $\bar{f}_\tau(w, \phi(y(\tau, y_1)) = y(\tau, y_1) - \phi(y(\tau, y_1))$.

Since the mapping $x_1 \to \phi(y(\tau, y_1))$ is a Lipschitz continuous mapping which is the identity mapping for $\tau = 0$, it follows from the Inverse Function Theorem that this mapping is open, and therefore it is a mapping of $\mathscr{S}(w)$ onto itself for small $\tau > 0$. By iteration, this is a mapping of $\mathscr{S}(w)$ onto itself for each $\tau > 0$.

We want to show that $\bar{f}_\tau \in Q$ for $T \leqslant \tau \leqslant 2T$. From (3.23) one obtains

$$|\bar{f}_\tau(w, x)| \leqslant K\varepsilon, \qquad (w, x) \in \mathscr{E},\ T \leqslant \tau \leqslant 2T. \tag{3.25}$$

From the definition of ϕ one has $f(w, x) \in \mathscr{N}(x)$ for all $(w, x) \in \mathscr{E}$. Hence $\bar{f}_\tau$ satisfies properties (2) and (3).

Since the mappings $(w, x_1) \to y_1, (w, x_1) \to y(\tau, y_1)$, and $(w, x_1) \to \phi(y(\tau, y_1))$ are continuous mappings of $\mathscr{E}$ into R^n, it follows that $\bar{f}_\tau$ is continuous on $\mathscr{E}$. (In fact, $\bar{f}_\tau(w, \phi(y(\tau, y_1)))$ is jointly continuous in (τ, w, x_1).) Since $f(w, x_1)$ is independent of w for $w \in \mathscr{S}(x_1)$, it follows that $y_1, y(\tau, y_1)$, $\phi(y(\tau, y_1))$, and $\bar{f}_\tau(w, \phi(y(\tau, y_1)))$ are also independent of w for $w \in \mathscr{S}(x_1)$. Hence $\bar{f}_\tau$ satisfies properties (1) and (5).

To complete the proof that $\bar{f}_\tau \in Q$, we need to verify that $\bar{f}_\tau$ satisfies the Lipschitz condition (4). Fix $w \in \mathscr{K}$ and let $x_0 \in \mathscr{S}(w)$, $x_1 \in \mathscr{D}(x_0)$, and $y_i = x_i + f(w, x_i)$, $i = 0, 1$. Let $\bar{y}_i = y(\tau, y_i)$ and $\bar{x}_i = \phi(y(\tau, y_i))$, $i = 0, 1$. From the definition of $\bar{f}_r$ it follows that

$$\bar{y}_i = \bar{x}_i + \bar{f}_\tau(w, \bar{x}_i), \qquad i = 0, 1.$$

Since the mapping $x_1 \to \bar{x}_1$ is open, there is an $\varepsilon_1(w, \bar{x}_1, \bar{f}_\tau)$, $0 < \varepsilon_1(w, \bar{x}_1, \bar{f}_\tau) \leqslant \varepsilon$, such that, for each $\bar{x}_0$ satisfying

$$|\bar{x}_1 - \bar{x}_0| \leqslant \varepsilon_1(w, \bar{x}_1, \bar{f}_\tau), \tag{3.26}$$

there is an $x_0 \in \mathscr{D}(x_1)$ satisfying $|x_1 - x_0| \leqslant \varepsilon_1(w, x_1, f) \leqslant \varepsilon$ and such that $\bar{x}_0 = \phi(y(\tau, y_0))$.

In order to estimate the difference $|\bar{f}_\tau(w, \bar{x}_1) - \bar{f}_\tau(w, \bar{x}_0)|$, we note that

$$\bar{y}_1 - \bar{y}_0 = (\bar{x}_0 - \bar{y}_0) + (\bar{x}_1 - \bar{x}_0) + (\bar{y}_1 - \bar{x}_1). \tag{3.27}$$

Let P_n be the orthogonal projection onto the space $\mathscr{U}^n(\bar{x}_0)$. Since $\bar{y}_0 \in \mathscr{N}(\bar{x}_0)$ one has $P_n(\bar{x}_0 - \bar{y}_0) = 0$. Consequently, it follows from (3.27) that

$$P_n(\bar{y}_1 - \bar{y}_0) = P_n(\bar{x}_1 - \bar{x}_0) + P_n(\bar{y}_1 - \bar{x}_1).$$

Since $\mathscr{K}$ is a hyperbolic attractor, $\mathscr{U}^n(\bar{x}_0)$ is the tangent space of $\mathscr{D}(\bar{x}_0)$ at the point $\bar{x}_0$, and since the angle $\angle(\mathscr{U}^n(\bar{x}_0), \bar{x}_1 - \bar{x}_0)$ is small, we have

$$|P_n(\bar{x}_1 - \bar{x}_0)| = |\bar{x}_1 - \bar{x}_0| + O(|\bar{x}_1 - \bar{x}_0|^2). \tag{3.28}$$

Likewise we find

$$|P_n(\bar{y}_1-\bar{x}_1)|=O(\varepsilon\,|\bar{x}_1-\bar{x}_0|). \tag{3.29}$$

From (3.26), (3.28), and (3.29) we obtain

$$|P_n(\bar{y}_1-\bar{y}_0)|=|\bar{x}_1-\bar{x}_0|+O(\varepsilon\,|\bar{x}_1-\bar{x}_0|).$$

Next let $P_{\mathcal{N}}$ be the orthogonal projection onto the space $\mathcal{N}(\bar{x}_0)$. From (3.27) we find

$$P_{\mathcal{N}}(\bar{y}_1-\bar{y}_0)=P_{\mathcal{N}}(\bar{x}_0-\bar{y}_0)+P_{\mathcal{N}}(\bar{x}_1-\bar{x}_0)+P_{\mathcal{N}}(\bar{y}_1-\bar{x}_1).$$

It is clear that

$$|P_{\mathcal{N}}(\bar{x}_1-\bar{x}_0)|=O(|\bar{x}_1-\bar{x}_0|^2). \tag{3.30}$$

By definition $\bar{f}_\tau(w,\bar{x}_0)=\bar{y}_0-\bar{x}_0$ and $\bar{f}_\tau(w,\bar{x}_0)\in\mathcal{N}(\bar{x}_0)$. Therefore

$$\begin{aligned}&P_{\mathcal{N}}(\bar{x}_0-\bar{y}_0)=-\bar{f}_\tau(w,\bar{x}_0)\\&P_{\mathcal{N}}(\bar{y}_1-\bar{x}_1)=P_{\mathcal{N}}\bar{f}_\tau(w,\bar{x}_1)=\bar{f}_\tau(w,\bar{x}_1)+O(\varepsilon\,|\bar{x}_1-\bar{x}_0|).\end{aligned} \tag{3.31}$$

From (3.26), (3.30), and (3.31) one obtains

$$|P_{\mathcal{N}}(\bar{y}_1-\bar{y}_0)|=|\bar{f}_\tau(w,\bar{x}_1)-\bar{f}_\tau(w,\bar{x}_0)|+O(\varepsilon\,|\bar{x}_1-\bar{x}_0|).$$

However

$$\frac{|P_{\mathcal{N}}(\bar{y}_1-\bar{y}_0)|}{|P_n(\bar{y}_1-\bar{y}_0)|}=\tan\left[\angle\,(\mathcal{U}^n(\bar{x}_0),\bar{y}_1-\bar{y}_0)\right].$$

Since $f\in Q$, it follows that

$$\angle\,(\mathcal{U}^n(x_1),y_1-y_0)\leqslant\tfrac{3}{4}c\alpha. \tag{3.32}$$

From (3.13), (3.19), and (3.32), we obtain

$$\angle\,(\mathcal{U}^n_Y(y_1),y_1-y_0)\leqslant c\alpha.$$

From (3.13), (3.19), and (3.22), we see that

$$\angle\,(\mathcal{U}^n(\bar{x}_0),\bar{y}_1-\bar{y}_0)\leqslant\tfrac{3}{4}c\alpha,\qquad T\leqslant\tau\leqslant 2T.$$

Consequently, we have

$$|\bar{f}_\tau(w,\bar{x}_1)-\bar{f}_\tau(w,\bar{x}_0)|\leqslant\tan\left(\tfrac{3}{4}c\alpha\right)|\bar{x}_1-\bar{x}_0|,\qquad T\leqslant\tau\leqslant 2T,$$

which completes the proof that $A_\tau Q\subset Q$, for $T\leqslant\tau\leqslant 2T$.

Since for every $y_0 \in U(\mathscr{K}, \bar{c}\beta)$, the linear system (3.16) is hyperbolic along $y(t, y_0)$ on the interval $0 \leqslant t \leqslant 2T$ with constants $3a$, $\lambda_1 - 2\sigma$, and $\lambda_2 + 2\sigma$, a straightforward adaptation of the arguments presented in Perron [11, 12] can be used to prove the following results: For every $f \in Q$ and for each $y_0 = x_0 + f(w, x_0)$, where $x_0 \in \mathscr{S}(w)$, there exists an $(n-k)$-dimensional Perron disk $\mathscr{D}^p_{Y,f}(y_0)$ such that if $y_1 \in \mathscr{D}^p_{Y,f}(y_0)$ then one has

$$| y(t, y_1) - y(t, y_0)| \leqslant 4a \,| y_1 - y_0 |\, e^{-(\lambda_1 - 3\sigma)t}, \qquad 0 \leqslant t \leqslant 2T.$$

We now repeat the argumentation leading to (3.14), but now we use it for the surface $y = x + f_1(w, x)$ over a given leaf $\mathscr{S}(w)$ in $\mathscr{K}$. In particular, for each $y_1 = x_1 + f_1(w, x_1)$, $x_1 \in \mathscr{S}(w) \subset \mathscr{K}$, on the surface $y = x + f_1(w, x)$, we use the fact that the Perron disks $\mathscr{D}^p_{Y,f_1}(y)$ form a foliation of a neighborhood $U(y_1, \beta/2)$ of y_1, and we construct a continuous mapping $\phi_{f_1} = \phi_{f_1, y_1}$ of $\Gamma_{f_1}(y_1, \beta/2)$ onto $f_1(\mathscr{D}(x_1))$. As argued for (3.14), one then obtains

$$| y(t, y_2) - \phi_{f_1}(y(t, y_2))| \leqslant | y(t, y_1) - y(t, y_2)| \leqslant 4a \,| y_1 - y_2 |\, e^{-(\lambda_1 - 3\sigma)t}, \tag{3.33}$$

for $0 \leqslant t \leqslant 2T$, provided $y_2 \in \mathscr{D}^p_{Y,f_1}(y_1)$. In particular, for $f_1, f_2 \in Q$ and $y_2 = f_2(w, x_1)$ by using (3.3) and (3.33), we obtain

$$| y(t, y_2) - \phi_{f_1}(y(t, y_2))| \leqslant \tfrac{3}{4} \,| y_1 - y_2 | = \tfrac{3}{4} \,| f_1(w, x_1) - f_2(w, x_1)| \leqslant \tfrac{3}{4} d(f_1, f_2),$$

provided that $T \leqslant t \leqslant 2T$. It follows then that

$$d(A_\tau f_1, A_\tau f_2) \leqslant \tfrac{3}{4} d(f_1, f_2), \qquad f_1, f_2 \in Q,\ T \leqslant \tau \leqslant 2T.$$

Thus the operator A_τ is contractive for $\tau \in [T, 2T]$, and therefore it has a fixed point $g_r \in Q$, i.e.,

$$A_\tau\, g_\tau = g_\tau, \qquad T \leqslant \tau \leqslant 2T.$$

Now set $g \stackrel{\text{def}}{=} g_T$.

Since A_τ is continuous in τ, it follows that g_τ is continuous in τ for $T \leqslant \tau \leqslant 2T$. Now for any rational number $\rho = p/q \in [0, 1]$ one has $A^q_{T+\rho T} = A^{p+q}_T$. Hence $g = g_{T+\rho T}$ for all rational $\rho \in [0, 1]$. By continuity, $g = g_\tau$, for $T \leqslant \tau \leqslant 2T$. For $0 < \tau < T$ one has $A_\tau\, g = A_\tau A_{T g} = A_{\tau + T g} = g$. Hence $A_\tau\, g = g$ for all $\tau > 0$.

Set $h(x) \stackrel{\text{def}}{=} x + g(x, x)$, $\mathscr{S}^Y = h(\mathscr{S})$, and $\mathscr{K}^Y = h(\mathscr{K})$. Since $A_\tau\, g = g$ for all $\tau > 0$, it follows that the sets $\mathscr{S}^Y$ and $\mathscr{K}^Y$ are invariant sets of the system (1.5). Furthermore (3.25) implies that $|h(x) - x| \leqslant K\varepsilon$ for all $x \in \mathscr{K}$. By rescaling ε we have $|h(x) - x| \leqslant \varepsilon$ for all $x \in \mathscr{K}$. Since the mapping $x \to g(x, x)$ is continuous on $\mathscr{K}$, it follows that h is continuous on $\mathscr{K}$.

Moreover, the restriction $h|_{\mathscr{S}}$ is Lipschitz continuous on the leaf $\mathscr{S}$. Also $h(x) \in \mathscr{N}(x) \cap \mathscr{K}^Y$ for all $x \in \mathscr{K}$. Note that for any $x \in \mathscr{K}$ one has

$$h(\phi(y(t, h(x)))=y(t, h(x)), \qquad t \in R. \tag{3.34}$$

It is easily seen that the mapping $(x, t) \to \hat{S}(t)\,x \stackrel{\text{def}}{=} \phi(y(t, h(x)))$ is continuous flow on $\mathscr{K}$; i.e., $\hat{S}(t)\,x$ is continuous in each variable, $\hat{S}(0)\,x = x$ for all $x \in \mathscr{K}$, and the group property $\hat{S}(\tau)\,\hat{S}(t)\,x = \hat{S}(\tau + t)\,x$ is valid. Finally (3.1) is simply a reformulation of (3.34). ∎

Remark. It is easily seen that the function $w(t) = \phi(y(t, h(x)))$ is a solution of the differential equation

$$\frac{dw}{dt} = \nabla\phi(h(w))[X(h(w)) + Y(h(w))]$$

satisfying $w(0) = x$, where $x \in \mathscr{K}$.

4. Hyperbolicity of $\mathscr{K}^{\mathscr{Y}}$

Let $\mathscr{S}$, $\mathscr{S}^Y$, $\mathscr{K}^Y$, and h have the same sense as in Theorem 1. Let $y_0 \in \mathscr{S}^Y$, $x_0 \in \mathscr{S}$, and $h(x_0) = y_0$. Let $w = \phi(y(t, h(x_0))$ be given by (3.34), and let $\Psi(t, y_0)$ be the fundamental operator solution of the linear system (3.16).

THEOREM 2. *For each $\sigma > 0$ there exists a $\delta > 0$ such that if* (1.6) *is satisfied with this δ, then the system* (3.16) *is hyperbolic along $y(t, y_0)$ on R with constants $(a + \sigma)$, $(\lambda_1 - \sigma)$, and $(\lambda_2 + \sigma)$ for every $y_0 \in \mathscr{K}^Y$. Let $\mathscr{U}^n_Y(y(t, y_0))$ and $\mathscr{U}^s_Y(y(t, y_0))$ denote the associated neutral and stable linear spaces for* (3.16). *Then $\mathscr{U}^n_Y(y(t, y_0))$ and $\mathscr{U}^s_Y(y(t, y_0))$ are arbitrarily close to the spaces $\mathscr{U}^n(\phi(y(t, y_0)))$ and $\mathscr{U}^s(\phi(y(t, y_0)))$, respectively, for all $t \in R$ provided δ is sufficiently small.*

The proof of this theorem is the same as the proof in Pliss [14, Theorem 1.3, Chap. 4, p. 257] so we omit the details.

THEOREM 3. *Let $y_0 = h(x_0)$ for $x_0 \in \mathscr{K}$. The space $\mathscr{U}^n_Y(y_0)$ is tangent to the set $h(\mathscr{D}(x_0))$ at the point y_0, provided ε and δ are sufficiently small.*

Proof. Assume on the contrary that there exists a sequence of points $x_i \in \mathscr{D}(x_0)$, $i = 1, 2, ...,$ such that $x_i \to x_0$, as $i \to \infty$, and for each i

$$\angle\,(\mathscr{U}^n_Y(y_0), h(x_i) - y_0) \geqslant \gamma_1 \tag{4.1}$$

for some $\gamma_1 > 0$. Let $y_i = h(x_i)$. From (1.8) and (1.9) (as applied to $\Psi(t, y_0)$) and (4.1) it follows that there exists a $T < 0$ such that

$$\angle\, (\mathscr{U}^s_Y(y(T, y_0)), \Psi(T, y_0)\, y_i) \leqslant 0.1\alpha, \tag{4.2}$$

where α is given by (3.2). From (4.2) and Theorem 2 it follows that for small δ one has

$$\angle\, (\mathscr{U}^s(\phi(y(T, y_0))), \Psi(T, y_0)\, y_i) \leqslant 0.2\alpha. \tag{4.3}$$

Since h is continuous on $\mathscr{D}(x_0)$ one has $y_i \to y_0$, as $i \to \infty$. Therefore there exists an i, say $i = j$, such that

$$|\, y(t, y_0) - y(t, y_j)| \leqslant \varepsilon$$

and

$$|\phi(\, y(t, y_0)) - \phi(\, y(t, y_j))| \leqslant \varepsilon$$

for all $t \in [T, 0]$.

From (4.3) it follows that for small δ one has

$$\angle\, (\mathscr{U}^s(\phi(\, y(T, y_0))), y(T, y_j) - y(T, y_0)) \leqslant 0.3\alpha. \tag{4.4}$$

From inequalities (3.2) and (4.4) we have

$$\angle\, (\mathscr{U}^n(\phi(\, y(T, y_0))), y(T, y_j) - y(T, y_0)) \geqslant 0.7\alpha. \tag{4.5}$$

Let $\bar{y}_i = y(T, y_i)$ and $\bar{x}_i = \phi(\, y(T, y_i))$, $i = 0, j$. We have

$$(\bar{y}_j - \bar{y}_0) = (\bar{x}_0 - \bar{y}_0) + (\bar{x}_j - \bar{x}_0) + (\,\bar{y}_j - \bar{x}_j). \tag{4.6}$$

As before let P_n and $P_{\mathscr{N}}$ be orthogonal projections onto the spaces $\mathscr{U}^n(\bar{x}_0)$ and $\mathscr{N}(\bar{x}_0)$, respectively. Since $P_n(\bar{x}_0 - \bar{y}_0) = 0$, it follows from (4.6) that

$$P_n(\,\bar{y}_j - \bar{y}_0) = P_n(\bar{x}_j - \bar{x}_0) + P_n(\,\bar{y}_j - \bar{x}_j). \tag{4.7}$$

Since $\mathscr{U}^n(x_0)$ is the tangent space of $\mathscr{D}(x_0)$ at the point x_0, one has

$$P_n(\bar{x}_j - \bar{x}_0) = |\,\bar{x}_j - \bar{x}_0| + O(|\,\bar{x}_j - \bar{x}_0|^2). \tag{4.8}$$

Likewise one has

$$|\,P_n(\,\bar{y}_j - \bar{x}_j)| = |\, g(\bar{x}_j, \bar{x}_j)| \cdot O(|\,\bar{x}_j - \bar{x}_0|),$$

where as before $g(x, x) = h(x) - x$. However, $|\, g(x, x)| \leqslant \varepsilon$, and therefore

$$|\,P_n(\,\bar{y}_j - \bar{x}_j)| = O(\varepsilon\, |\,\bar{x}_j - \bar{x}_0|). \tag{4.9}$$

From (4.7), (4.8), and (4.9) we have the estimate

$$|P_n(\bar{y}_j - \bar{y}_0)| \leqslant |\bar{x}_j - \bar{x}_0| + O(\varepsilon\, |\bar{x}_j - \bar{x}_0|). \tag{4.10}$$

Similarly from (4.6) we have

$$P_{\mathcal{N}}(\bar{y}_j - \bar{y}_0) = P_{\mathcal{N}}(\bar{x}_0 - \bar{y}_0) + P_{\mathcal{N}}(\bar{x}_j - \bar{x}_0) + P_{\mathcal{N}}(\bar{y}_j - \bar{x}_j). \tag{4.11}$$

It is clear that

$$|P_{\mathcal{N}}(\bar{x}_j - \bar{x}_0)| = O(|\bar{x}_j - \bar{x}_0|^2). \tag{4.12}$$

Since $g(\bar{x}_i, \bar{x}_i) = \bar{y}_i - \bar{x}_i \in \mathcal{N}(\bar{x}_i)$, $i = 0, j$, we have

$$P_{\mathcal{N}}(\bar{x}_0 - \bar{y}_0) = -g(\bar{x}_0, \bar{x}_0), \tag{4.13}$$

and

$$P_{\mathcal{N}}(\bar{y}_j - \bar{x}_j) = g(\bar{x}_j, \bar{x}_j) + O(\varepsilon\, |\bar{x}_j - \bar{x}_0|). \tag{4.14}$$

From (4.11)–(4.14) we obtain

$$|P_{\mathcal{N}}(\bar{y}_j - \bar{y}_0)| = |g(\bar{x}_j, \bar{x}_j) - g(\bar{x}_0, \bar{x}_0)| + O(\varepsilon\, |\bar{x}_j - \bar{x}_0|). \tag{4.15}$$

From (4.10) and (4.15) we obtain

$$\tan[\angle\, (\mathcal{U}^n(\bar{x}_0), \bar{y}_j - \bar{y}_0))] = \frac{|P_{\mathcal{N}}(\bar{y}_j - \bar{y}_0)|}{|P_n(\bar{y}_j - \bar{y}_0)|} = \frac{|g(\bar{x}_j, \bar{x}_j) - g(\bar{x}_0, \bar{x}_0)|}{|\bar{x}_j - \bar{x}_0|} + O(\varepsilon).$$

From Section 3 we obtain

$$\tan[\angle\, (\mathcal{U}^n(\bar{x}_0), \bar{y}_j - \bar{y}_0))] \leqslant \tan(\tfrac{3}{4}c\alpha) + O(\varepsilon),$$

which contradicts (4.5) for small ε, since $c \leqslant 1/2$. ∎

5. One-to-One Property

We conclude the proof by showing that for ε sufficiently small, h is a homeomorphism.

Theorem 4. *Let the hypotheses of Theorem 1 be satisfied. Then for sufficiently small ε the mapping $h\colon \mathcal{K} - \mathcal{K}^Y$ is one to one.*

Proof. As in Section 3, we fix $\beta > 0$ such that in the β-neighborhood $U(x(t, x_0), \beta)$ of the point $x(t, x_0)$ the linear system (1.7) dominates the nonlinear system (1.4). In particular, these exists a T (which does not depend on x_0) such that $T<0$ and if points x_0 and x_1 satisfy $\angle(\mathcal{N}(x_0), x_1-x_0)\leqslant\pi/4$ and $|x(t, x_1)-x(t, x_0)|\leqslant\beta$, for $T\leqslant t\leqslant 0$, then

$$|x(T, x_1)-x(T, x_0)|\geqslant\frac{12}{\sin\alpha}|x_1-x_0| \tag{5.1}$$

and

$$\angle(\mathcal{U}^n(x(T, x_0)), x(T, x_1)-x(T, x_0))\geqslant\frac{3\alpha}{4}, \tag{5.2}$$

where α is given by inequality (3.2).

Choose the radius r of the disks $\mathcal{D}(x_0)$ given in the definition of a hyperbolic attractor small enough so that the following two inequalities hold:

$$r\leqslant\min\left\{\frac{\ln 2}{L}, \frac{1}{2L}\sin\frac{\alpha}{2}, \frac{\beta}{2}\right\}, \tag{5.3}$$

where L is the Lipschitz constant of the mapping $x_0\to\mathcal{U}^n(x_0)$, and

$$\angle(\mathcal{U}^n(x_0), \mathcal{U}^n(x_1))\leqslant\frac{\alpha}{4}, \qquad x_0\in\mathcal{K}, x_1\in\mathcal{D}(x_0). \tag{5.4}$$

It follows from the continuity of solutions that there exists a $\mu\in(0, 1/3]$ such that

$$U(x(t, x_0), \mu r)\subset x(t, U(x_0, r)), \qquad \text{for} \quad T\leqslant t\leqslant 0, \text{and all } x_0\in\mathcal{K}.$$

Take the numbers ε and δ so small that the following condition is satisfied: If (1.6) is satisfied with this δ, $|y_0-x_0|\leqslant 2\varepsilon$, and $x_0\in\mathcal{K}$, then

$$|x(t, x_0)-y(t, y_0)|\leqslant\frac{\mu r}{3}, \qquad \text{for} \quad T\leqslant t\leqslant 0. \tag{5.5}$$

Assume, in contradiction of the statement of the theorem, that there exist two points $x_0\in\mathcal{K}$ and $x_1\in\mathcal{K}$ such that $x_0\neq x_1$ and $h(x_0)=h(x_1)=y_0$. Since $|y_0-x_0|\leqslant\varepsilon$ and $|y_0-x_1|\leqslant\varepsilon$, one has $|x_0-x_1|\leqslant 2\varepsilon$. We will use this and the fact that ε is small.

Our first objective is to show that the angles

$$\rho\overset{\text{def}}{=}\angle(\mathcal{N}(x_0), x_1-x_0), \qquad \gamma\overset{\text{def}}{=}\angle(y_0-x_0, y_0-x_1)$$

are small. From the definition of h it follows that $(y_i-x_i)\in\mathcal{N}(x_i)$, $i=0, 1$. Consequently, it follows from the Lipschitz property that one has $\tan\gamma\leqslant L|x_1-x_0|$. There now exist two cases:

(1) $|y_0 - x_i| \geqslant \max(|y_0 - x_j|, |x_1 - x_0|)$, where $i \neq j$ and $i, j \in \{0, 1\}$.

(2) $|x_1 - x_0| \geqslant \max(|y_0 - x_1|, |y_0 - x_0|)$.

In case (1) there is no loss in generality in assuming $i = 0$ and $j = 1$. One then has $\gamma \geqslant \sin\gamma \geqslant (1/2)\,|x_1 - x_0|\,|x_1 - y_0|^{-1}\sin\rho$. Therefore $(1/2)\,|x_1 - x_0|\,|x_1 - y_0|^{-1}\sin\rho \leqslant \arctan(L\,|x_1 - x_0|)$. For small ε, the last inequality becomes $\sin\rho \leqslant 2L\varepsilon + O(\varepsilon^2)$. In case (2) one has $\rho \leqslant \gamma \leqslant \arctan 2L\varepsilon$. In either case, one can ensure that both ρ and γ are small by choosing ε small. In what follows we assume that ε is so small that

$$\angle\,(\mathcal{N}(x_0), x_1 - x_0) < \frac{\pi}{4}.$$

For $x \in \mathcal{K}$ let $d(x)$ be the k-dimensional disk with the center at x, radius μr, and such that $d(x) \subset \mathcal{D}(x)$. Using the facts that the mapping h is continuous and $\phi(y(t, h(x_i))) \in x(t, \mathcal{D}(x_i))$, for $i = 0, 1$ and $T \leqslant t \leqslant 0$, together with (5.5), for small ε one obtains

$$\phi(y(t, h(x_i))) \in d(x(t, x_i)) \qquad i = 0, 1,\ T \leqslant t \leqslant 0. \tag{5.6}$$

Let $\bar{x}_i = x(T, x_i)$ and $x_i^{(1)} = \phi(y(T, h(x_i)))$, $i = 0, 1$. From (5.6) it follows that $x_i^{(1)} \in d(\bar{x}_i)$, therefore $\bar{x}_i \in d(x_i^{(1)}) \subset \mathcal{D}(x_i^{(1)})$, $i = 0, 1$. Let us transform the origin into the point $x_0^{(1)}$ and rotate the coordinate axes to obtain the new coordinate system (u, v), where u is a k-dimensional vector and v is an $(n-k)$-dimensional vector. We assume that in the new coordinate system the space $v = 0$ coincides with the linear space $\mathcal{U}^n(x_0^{(1)})$. In these coordinates the disks $\mathcal{D}(x_0^{(1)})$ and $\mathcal{D}(x_1^{(1)})$ can be represented in the form

$$\mathcal{D}(x_i^{(1)}) = \{v = f_i(u) : |u| \leqslant r\}, \qquad i = 0, 1,$$

where f_i are continuously differentiable $(n-k)$-dimensional vector-valued functions, $i = 0, 1$, and $f_0(0) = 0$. In this coordinate system the points $\bar{x}_i$ have representations $(u_i, f_i(u_i))$, $i = 0, 1$. Let $\bar{x}_2 = (u_1, f_0(u_1))$. Consider vectors $z_0 = \bar{x}_0 - \bar{x}_2$, $z_1 = \bar{x}_1 - \bar{x}_0$, $z_2 = \bar{x}_1 - \bar{x}_2$. It is clear that $z_2 = z_0 + z_1$. Let P be the projector onto the space $\{(0, v)\}$. We then have $Pz_2 = Pz_0 + Pz_1$ and

$$|Pz_2| \geqslant |Pz_1| - |Pz_0|. \tag{5.7}$$

From (5.2) it follows that $\angle\,(\mathcal{U}^n(\bar{x}_0), z_1) \geqslant 3\alpha/4$. By using the additivity of the angles and (5.4) we find $\angle\,(\mathcal{U}^n(x_0^{(1)}), z_1) \geqslant \alpha/2$. From this inequality it follows that

$$|Pz_1| \geqslant |z_1| \sin\frac{\alpha}{2}. \tag{5.8}$$

Let us estimate $|Pz_0|$. Since $Pz_0=(0, f_0(u_0)-f_0(u_1))$ we get

$$|Pz_0|=|f_0(u_0)-f_0(u_1)|.$$

Now

$$|Pz_0|=|f_0(u_0)-f_0(u_1)|\leqslant \max_{|u|\leqslant r}\left|\frac{\partial f_0}{\partial u}\right| |u_1-u_0|\leqslant Lr\,|z_1|,$$

since the Lipschitz property implies that $|\partial f_0/\partial u|\leqslant Lr$ for $|u|\leqslant r$. From this inequality, (5.7), and (5.8) it follows that $|Pz_2|\geqslant(\sin(\alpha/2)-Lr)|z_1|$, and (5.3) implies that

$$|Pz_2|\geqslant\frac{1}{2}\sin\frac{\alpha}{2}\,|z_1|,$$

which can be rewritten in the form

$$|f_1(u_1)-f_0(u_1)|\geqslant\frac{1}{2}\sin\frac{\alpha}{2}\,|x(T, x_1)-x(T, x_0)|.$$

Using (5.1) we then find

$$|f_1(u_1)-f_0(u_1)|\geqslant 6\,|x_1-x_0|. \tag{5.9}$$

Let $w(s)=f_1((1-s)\,u_1)-f_0((1-s)\,u_1)$, $0\leqslant s\leqslant 1$. Then

$$\frac{dw}{ds}=\left(\frac{\partial f_0}{\partial u}-\frac{\partial f_1}{\partial u}\right)u_1,$$

which, together with the Lipschitz property, implies that

$$\left|\frac{d\,|w|}{ds}\right|\leqslant\left|\frac{dw}{ds}\right|\leqslant\left|\frac{\partial f_0}{\partial u}-\frac{\partial f_1}{\partial u}\right| |u_1|\leqslant L\,|w(s)|\,|u_1|\leqslant Lr\,|w(s)|,$$

since $|u_1|\leqslant r$. By using (5.3) we have

$$\frac{d\,|w|}{ds}\geqslant -\ln 2\,|w|.$$

From the Gronwall inequality we obtain $2\,|w(1)|\geqslant|w(0)|$, or equivalently,

$$|f_1(0)-f_0(0)|\geqslant\tfrac{1}{2}\,|f_1(u_1)-f_0(u_1)|.$$

Combining this with (5.9) and using the fact that $f_0(0)=0$, we obtain

$$|f_1(0)|\geqslant 3\,|x_1-x_0|. \tag{5.10}$$

In the (u, v) coordinates $x_0^{(1)}=(0,0)$, and for some u_2 with $|u_2|\leqslant r/3$, one has $x_1^{(1)}=(u_2, f_1(u_2))$. Therefore

$$|x_1^{(1)}-x_0^{(1)}|^2=|u_2|^2+|f_1(u_2)|^2. \tag{5.11}$$

Since $|u_2|\leqslant r/3$, we have the inequality

$$|f_1(u_2)|\geqslant -\max_{|u|\leqslant|u_2|}\left|\frac{\partial f_1(u)}{\partial u}\right||u_2|+|f_1(0)|\geqslant|f_1(0)|-\frac{Lr}{3}|u_2|.$$

As a result, (5.11) implies that

$$|x_1^{(1)}-x_0^{(1)}|^2\geqslant|u_2|^2+\left(|f_1(0)|-\frac{Lr}{3}|u_2|\right)^2.$$

By expanding and using the Young inequality we get

$$|x_1^{(1)}-x_0^{(1)}|^2\geqslant\left(1-\frac{4}{45}L^2r^2\right)|u_2|^2+\frac{4}{9}|f_1(0)|^2,$$

which together with (5.3) and (5.10) implies that

$$|x_1^{(1)}-x_0^{(1)}|\geqslant\tfrac{2}{3}|f_1(0)|\geqslant 2|x_1-x_0|. \tag{5.12}$$

Let $y_0^{(1)}=y(T, y_0)$. Since $h(x_0)=h(x_1)=y_0$, it follows from (3.1) and (3.34) that $h(x_0^{(1)})=h(x_1^{(1)})=y_0^{(1)}$. Since $|h(x)-x|\leqslant\varepsilon$, we have $|x_0^{(1)}-y_0^{(1)}|\leqslant\varepsilon$ and $|x_1^{(1)}-y_0^{(1)}|\leqslant\varepsilon$.

Using the same considerations for the points $x_0^{(1)}$, $x_1^{(1)}$, and $y_0^{(1)}$ we obtain points $x_0^{(2)}$, $x_1^{(2)}$, and $y_0^{(2)}$ with the same properties, but (5.12) yields

$$|x_1^{(2)}-x_0^{(2)}|\geqslant 2|x_1^{(1)}-x_0^{(1)}|\geqslant 2^2|x_1-x_0|.$$

Continuing this process we will obtain a sequence of points $x_0^{(m)}$, $x_1^{(m)}$, $y_0^{(m)}$ lying in a bounded set with the properties $|x_0^{(m)}-y_0^{(m)}|\leqslant\varepsilon$, $|x_1^{(m)}-y_0^{(m)}|\leqslant\varepsilon$, and

$$|x_1^{(m)}-x_0^{(m)}|\geqslant 2^m|x_1-x_0|,$$

which is impossible for $x_1\neq x_0$. The contradiction proves the theorem. ∎

Remark. Since $h(x)\in\mathcal{N}(x)\cap\mathcal{S}^Y$ for $x\in\mathcal{S}$ and $|h(x)-x|\leqslant\beta$, the inverse h^{-1} is given by $h^{-1}(y)=\phi(y)$ for any $y\in\mathcal{S}^Y\subset\mathcal{K}^Y$.

6. EXAMPLES

A stable fixed point and a stable periodic orbit are hyperbolic attractors that satisfy the Lipschitz property. More generally, a smooth, stable, normally hyperbolic, invariant manifold is a hyperbolic attractor that satisfies the Lipschitz property. The fact that a normally hyperbolic, invariant manifold persists under a small C^1-perturbation is, of course, well known. See for example, Fenichel [3], Hirsch, Pugh, and Shub [7], and Sacker [15]. In this section we give several examples of differential equations with hyperbolic attractors that satisfy the Lipschitz property. These examples include:

(1) a set $\mathscr{K}$ which is not a manifold, and

(2) a set $\mathscr{K}$ which is an invariant manifold, but which is not normally hyperbolic.

Before doing this though, we note that the theory described above extends readily to the study of diffeomorphism, or Poincaré mappings. Since a small perturbation of a diffeomorphism preserves a time synchronization, the theory of diffeomorphisms is somewhat simpler than that of differential equations. For example, in the case of perturbations of diffeomorphisms the induced flow $\hat{S}(t)\, x_0$ on $\mathscr{K}$, as described at the end of Section 3, agrees with the original flow $x(t, x_0)$.

6.1. *Attractors for Diffeomorphisms*

Let A be a 2×2 matrix with integer entries, det $A = 1$, and distinct real eigenvalues. Denote the eigenvalues by μ_1, μ_2. One then has $\mu_1 \mu_2 = 1$. We assume that $0 < \mu_2 < 1 < \mu_1$. For example, let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}.$$

The linear mapping $u \to Au$ preserves the integer lattice Z^2 in R^2, and therefore this mapping induces a diffeomorphism $u \to F(u)$ on the 2-dimensional torus T^2. This diffeomorphism is referred to as an Anosov diffeomorphism; see Arnold [1] and Smale [18].

Fix μ, $0 < \mu < 1$. On the product space $T^2 \times (-1, 1)$, we introduce the diffeomorphism

$$(u, v) \to (F(u), \mu v). \tag{6.1}$$

Then $M = T^2 \times \{0\}$ is a hyperbolic attractor for (6.1) that satisfies the Lipschitz property. For each $(u, 0) \in M$, the leaf $\mathscr{S}(u, 0)$ is the unstable manifold through $(u, 0)$, and $\dim \mathscr{S}(u, 0) = 1$. The fact that M persists under small C^1-perturbations is a consequence of the classical theory of structural stability; see Smale [18].

6.2. *Suspensions of Diffeomorphisms*

The use of the suspension method allows any diffeomorphism to be imbedded into a C^1-flow; see Smale [18]. For example, the suspension of the diffeomorphism $u \to F(u)$ on the torus T^2 described above gives rise to a C^1-flow $\pi(t, x)$ on a 3-dimensional manifold M^3. This is an example of an Anosov flow; see Arnold [1]. From the Whitney Imbedding Theorem, M^3 can be imbedded smoothly into the Euclidean space R^7, and the image of M^3 has a collared neighborhood U, which can be described in local coordinates as (x, z), where $x \in M^3$ and z belongs to a 4-dimensional disk $\hat{\mathscr{D}}(x)$. Let ν be a given positive number. By introducing the flow

$$(x, z) \to (\pi(t, x), ze^{-\nu t}), \qquad t \in R,$$

in this neighborhood, we see that M^3 becomes an attractor for an ordinary differential equation $X' = G(X)$ on U. Furthermore, for all $\nu > 0$, M^3 is a hyperbolic attractor that satisfies the Lipschitz property, and the associated disks $\mathscr{D}(x, 0)$ are two dimensional. However, M^3 is a normally hyperbolic manifold for $X' = G(X)$ if and only if $\nu > \ln \mu_1 > 0$; see Hirsch, Pugh, and Shub [7], for example.

5.3. *The Looking-Glass Attractor*

The next example is a hyperbolic attractor $\mathscr{K}$ for a C^∞-flow in the plane R^2; see Fig. 1. This attractor is not a manifold. We let Γ_1 and Γ_2 denote two stable periodic orbits in R^2, and we let P denote a hyperbolic fixed point which lies outside both Γ_1 and Γ_2. Assume that the two branches $\mathscr{W}_i^u(P)$, $i = 1, 2$ of the unstable manifold of P have Γ_1 and Γ_2 as their respective limits. Then

$$\mathscr{K} = \{P\} \cup \mathscr{W}_1^u(P) \cup \mathscr{W}_2^u(P) \cup \Gamma_1 \cup \Gamma_2$$

is a hyperbolic attractor that satisfies the Lipschitz property.

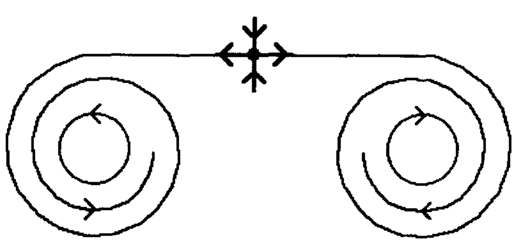

FIG. 1. The looking glass attractor.

6.4. *Products of Attractors*

The k-dimensional torus T^k can be imbedded into an open set Ω in $R^{k+\ell}$ by using the coordinate system

$$(\theta; \sigma) = (\theta_1, ..., \theta_k; \sigma_1, ..., \sigma_\ell), \qquad 0 \leqslant \theta_i < 2\pi, \sigma = (\sigma_1, ..., \sigma_\ell) \in \mathscr{D}^\ell,$$

where $\mathscr{D}^\ell$ denotes an ℓ-dimensional disk.

Let $\mathscr{K}_0$ denote a given hyperbolic attractor with constants a, λ_1, and λ_2 for an ordinary differential equation $x' = X(x)$ on R^n. Then $\mathscr{K}_1 = \mathscr{K}_0 \times T^k$ is a hyperbolic attractor for the product flow

$$\begin{aligned} x' &= F(x, \theta, \sigma) \\ \theta' &= \Theta(x, \theta, \sigma) \\ \sigma' &= -A\sigma + G(x, \theta, \sigma) \end{aligned} \tag{6.2}$$

on $R^n \times \Omega$, provided A is an $(\ell \times \ell)$ matrix whose eigenvalues μ_i satisfy $\operatorname{Re} \mu_i > \max(0, \lambda_2)$ and F, Θ, and G are C^1-functions that satisfy $F(x, \theta, 0) = X(x)$, $\Theta(x, \theta, 0) = \omega$, $G(x, \theta, 0) = 0$, and $D_3 G(x, \theta, 0) = 0$, where $D_3 = \partial/\partial z$ and $\omega = (\omega_1, ..., \omega_k)$ is a constant vector. If $\mathscr{K}_0$ satisfies the Lipschitz property, then so does $\mathscr{K}_1$.

Let $\mathscr{D}_i(x_0)$, $x_0 \in \mathscr{K}_0$, denote the disks in $\mathscr{K}_i$, $i = 0, 1$, as prescribed in the definition of a hyperbolic attractor. Let dim $\mathscr{D}_0 = k_0$; then one has $\dim \mathscr{D}_1 = k_0 + k$. If there is an eigenvalue μ_i that satisfies $\max(0, \lambda_2) < \operatorname{Re} \mu_i \leqslant \lambda_1$, then the attractor $\mathscr{K}_1$ is not normally hyperbolic; see Hirsch, Pugh, and Shub [7]. Even through $\mathscr{K}_1$ need not be normally hyperbolic, as the in the last two examples, our Main Theorem is applicable to perturbations of the underlying equation.

Remark. It is instructive to compare our theory as it applies to (6.2) with the examples constructed in Fenichel [3] of perturbations of invariant manifolds which are not normally hyperbolic. To be specific assume that $\mathscr{K}_0$ is the hyperbolic attractor in R^7 constructed in Section 6.2 with $0 < \nu \leqslant \ln \mu_1$. Our Main Theorem asserts that if one adds any small C^1-perturbation term Y to (6.2), then there is a homeomorphism h of $\mathscr{K}_1$ onto a hyperbolic attractor $\mathscr{K}^Y$ for the perturbed equation. The Fenichel construction suggests that since $\mathscr{K}_1$ is not normally hyperbolic, there exists a small perturbation such that the perturbed equation does not have a *smooth* invariant manifold near $\mathscr{K}_1$. If such a case, we conclude that, while $h: \mathscr{K}_1 \to \mathscr{K}^Y$ and $h^{-1}: \mathscr{K}^Y \to \mathscr{K}_1$ are both continuous, at least one of them is not differentiable. As we have shown in our theory, for each leaf $\mathscr{S} \subset \mathscr{K}_1$, the restriction $h|_{\mathscr{S}}$ and its inverse are locally Lipschitz continuous. It follows that a breakdown in smoothness can only occur in the direction normal to $\mathscr{S}$. Wrinkles can only arise in the stable direction!

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